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The equivalence between the convergences of Ishikawa and Mann iterations for an asymptotically pseudocontractive map

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Abstract

The convergence of Mann iteration is equivalent to the convergence of Ishikawa iterations, when T is an asymptotically nonexpansive and asymptotically pseudocontractive map.

Keywords: Mann type iteration; Ishikawa type iteration; Asymptotically nonexpansive; Asymptotically pseudocontractive

1. Introduction

Let X be a real Banach space and let B be a nonempty, convex subset. Let $u_1, x_1 \in B$ be two arbitrary fixed points. Let $T: B \to B$ be a map.

Definition 1. The map T is said to be *asymptotically nonexpansive* if there exists a sequence $(k_n)_n$, $k_n \in [1, \infty)$, $\forall n \in \mathbb{N}$, $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|, \quad \forall x, y \in B, \ \forall n \in \mathbb{N}. \tag{1}$$

The following remark will be useful.

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Remark 2. An asymptotically nonexpansive map is uniformly Lipschitzian for some $L \geqslant 1$, i.e., $(\exists L \geqslant 1\colon\ \|T^n x - T^n y\| \leqslant L\|x-y\|,\ \forall x, y \in B,\ \forall n \in \mathbb{N})$.

Proof. Let $L := \sup_{n\in\mathbb{N}} k_n$. Because $\lim_{n\to\infty} k_n = 1$ and $k_n \geqslant 1$, $\forall n \in \mathbb{N}$, one can deduce that $L \in [1, \infty)$. From Definition 1,

$$\|T^n x - T^n y\| \leqslant k_n\|x-y\| \leqslant L\|x-y\|, \quad \forall x, y \in B,\ \forall n \in \mathbb{N}. \qquad \square$$

In [5] the following class of maps was introduced.

Definition 3. A map T is said to be *asymptotically pseudocontractive* if there exists a sequence $(k_n)_n$, $k_n \in [1, \infty)$, $\forall n \in \mathbb{N}$, $\lim_{n\to\infty} k_n = 1$, and there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leqslant k_n\|x-y\|^2, \quad \forall x, y \in B,\ \forall n \in \mathbb{N}. \tag{2}$$

When $n = 1$ in (2) we get the usual definition of a strongly pseudocontractive map. The following remark is Remark 1 from [5].

Remark 4 [5]. An asymptotically nonexpansive map is *asymptotically pseudocontractive*. The converse is not true.

We consider the following iteration (see [3]):

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T^n u_n. \tag{3}$$

The sequence $(\alpha_n)_n \subset (0,1)$ is convergent, such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. This iteration is known as *Mann type iteration*. We consider the following iteration, known as *Ishikawa type iteration* (see [1]):

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \quad n = 1, 2, \ldots. \end{aligned} \tag{4}$$

The sequences $(\alpha_n)_n, (\beta_n)_n \subset (0,1)$ are such that

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \beta_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{5}$$

The sequence $(\alpha_n)_n$ remains the same in both iterations. For $\beta_n = 0$, $\forall n \in \mathbb{N}$, from (5) we get (4). We denote by $F(T) = \{x^* \in B\colon\ F(x^*) = x^*\}$. Replacing T^n by T in (3) and (5) gives the *Mann* and *Ishikawa iteration*, respectively.

The aim of this paper is to prove an equivalence between the convergences of the above two iterations when T is an asymptotically nonexpansive respective asymptotically pseudocontractive map.

The following lemma appears in [2].

Lemma 5 [2]. *Let X be a Banach space and $x, y \in X$. Then $\|x\| \leqslant \|x + ry\|$ for all $r > 0$ if and only if there exists $j(x) \in J(x)$ such that $\langle y, j(x)\rangle \geqslant 0$.*

Using this lemma we are able to prove the following result.

Lemma 6. *Let B be a nonempty subset of a Banach space X and let $T : B \to B$ be a map. Then the following conditions are equivalent*:

(i) *T is asymptotically pseudocontractive map*;
(ii) *There exists $k_n \in [1, \infty)$ such that*

$$\|x - y\| \leqslant \bigl\|x - y + r\bigl[(k_n I - T^n)x - (k_n I - T^n)y\bigr]\bigr\|, \quad \forall x, y \in B,\ r > 0. \tag{6}$$

Proof. Lemma 5 assures that relation (6) is $\langle (k_n I - T^n)x - (k_n I - T^n)y, j(x - y)\rangle \geqslant 0$, $\forall n \geqslant n_0$, which is equivalent with $\langle T^n x - T^n y, j(x - y)\rangle \leqslant k_n\langle x - y, j(x - y)\rangle = k_n\|x - y\|^2$, $\forall x, y \in B$, that is (2). □

The following lemma is Lemma 4 from [6].

Lemma 7 [6]. *Let $(a_n)_n$ be a nonnegative sequence which satisfies the following inequality*:

$$a_{n+1} \leqslant (1 - \lambda_n)a_n + \sigma_n, \tag{7}$$

where $\lambda_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

We are now able to give the following result.

Theorem 8. *Let B be a closed convex subset of an arbitrary Banach space X and $(x_n)_n$ and $(u_n)_n$ defined by* (3) *and* (4) *with $(\alpha_n)_n$ and $(\beta_n)_n$ satisfying* (5). *Let T be an asymptotically pseudocontractive and Lipschitzian with $L \geqslant 1$ self-map of B. Let x^* be the fixed point of T. If $u_0 = x_0 \in B$, then the following two assertions are equivalent*:

(i) *Mann type iteration* (3) *converges to $x^* \in F(T)$*;
(ii) *Ishikawa type iteration* (4) *converges to $x^* \in F(T)$.*

Proof. If the Ishikawa iteration (4) converges then setting $\beta_n = 0$, $\forall n \in \mathbb{N}$, the convergence of Mann iteration (3). Conversely, we shall prove that (i) $\Rightarrow$ (ii). The proof is similar to the proof of Theorem 4 from [4]. We have

$$\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T^n y_n \\
&= \bigl(1 + \alpha_n^2\bigr)x_{n+1} + \alpha_n(\alpha_n k_n I - T^n)x_{n+1} \\
&\quad - (1 + k_n)\alpha_n^2 x_{n+1} + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n) \\
&= \bigl(1 + \alpha_n^2\bigr)x_{n+1} + \alpha_n(\alpha_n k_n I - T^n)x_{n+1} \\
&\quad - (1 + k_n)\alpha_n^2\bigl[x_n + \alpha_n(T^n y_n - x_n)\bigr] + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n)
\end{aligned}$$

$$
\begin{aligned}
&= \left(1+\alpha_n^2\right)x_{n+1} + \alpha_n(\alpha_n k_n I - T^n)x_{n+1} - (1+k_n)\alpha_n^2 x_n \\
&\quad + (1+k_n)\alpha_n^3(x_n - T^n y_n) + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n) \\
&= \left(1+\alpha_n^2\right)x_{n+1} + \alpha_n(\alpha_n k_n I - T^n)x_{n+1} + \left[1-(1+k_n)\alpha_n\right]\alpha_n x_n \\
&\quad + (1+k_n)\alpha_n^3(x_n - T^n y_n) + \alpha_n(T^n x_{n+1} - T^n y_n). \qquad (8)
\end{aligned}
$$

Also

$$
\begin{aligned}
u_n &= u_{n+1} + \alpha_n u_n - \alpha_n T^n u_n \\
&= \left(1+\alpha_n^2\right)u_{n+1} + \alpha_n(\alpha_n k_n I - T^n)u_{n+1} \\
&\quad - (1+k_n)\alpha_n^2 u_{n+1} + \alpha_n u_n + \alpha_n(T^n u_{n+1} - T^n u_n) \\
&= \left(1+\alpha_n^2\right)u_{n+1} + \alpha_n(\alpha_n k_n I - T^n)u_{n+1} \\
&\quad - (1+k_n)\alpha_n^2\left[u_n + \alpha_n(T^n u_n - u_n)\right] + \alpha_n u_n + \alpha_n(T^n u_{n+1} - T^n u_n) \\
&= \left(1+\alpha_n^2\right)u_{n+1} + \alpha_n(\alpha_n k_n I - T^n)u_{n+1} + (1+k_n)\alpha_n^3(u_n - T^n u_n) \\
&\quad + \left[1-(1+k_n)\alpha_n\right]\alpha_n u_n + \alpha_n(T^n u_{n+1} - T^n u_n). \qquad (9)
\end{aligned}
$$

From (8) and (9) we get

$$
\begin{aligned}
x_n - u_n = {} & \left(1+\alpha_n^2\right)(x_{n+1} - u_{n+1}) + \alpha_n\left((\alpha_n k_n I - T^n)x_{n+1} - (\alpha_n k_n I - T^n)u_{n+1}\right) \\
& + \left[1-(1+k_n)\alpha_n\right]\alpha_n(x_n - u_n) + (1+k_n)\alpha_n^3(x_n - u_n - T^n y_n + T^n u_n) \\
& + \alpha_n(T^n x_{n+1} - T^n u_{n+1} - T^n y_n + T^n u_n). \qquad (10)
\end{aligned}
$$

The norm of the sum of the first two terms on the right-hand side of (10) is equal to

$$
\left(1+\alpha_n^2\right)\left\|(x_{n+1} - u_{n+1}) + \frac{\alpha_n}{1+\alpha_n^2}\left((\alpha_n k_n I - T^n)x_{n+1} - (\alpha_n k_n I - T^n)u_{n+1}\right)\right\|.
$$

Using (6) with $x := x_{n+1}$, $y := u_{n+1}$, we obtain

$$
\begin{aligned}
&\left\|\left(1+\alpha_n^2\right)(x_{n+1} - u_{n+1}) + \alpha_n\left((\alpha_n k_n I - T^n)x_{n+1} - (\alpha_n k_n I - T^n)u_{n+1}\right)\right\| \\
&\quad \geqslant \left(1+\alpha_n^2\right)\|x_{n+1} - u_{n+1}\|. \qquad (11)
\end{aligned}
$$

From (10) it follows that

$$
\begin{aligned}
\|x_n - u_n\| \geqslant {} & \Big\|\left(1+\alpha_n^2\right)(x_{n+1} - u_{n+1}) \\
& + \alpha_n\left((\alpha_n k_n I - T^n)x_{n+1} - (\alpha_n k_n I - T^n)u_{n+1}\right)\Big\| \\
& + \left[1-(1+k_n)\alpha_n\right]\alpha_n\|x_n - u_n\| \\
& - (1+k_n)\alpha_n^3\|x_n - u_n - T^n y_n + T^n u_n\| \\
& - \alpha_n\|T^n x_{n+1} - T^n u_{n+1} - T^n y_n + T^n u_n\| \\
\geqslant {} & (1+\alpha_n)\|x_{n+1} - u_{n+1}\| + \left[1-(1+k_n)\alpha_n\right]\alpha_n\|x_n - u_n\| \\
& - (1+k_n)\alpha_n^3\|x_n - u_n - T^n y_n + T^n u_n\| \\
& - \alpha_n\|T^n x_{n+1} - T^n u_{n+1} - T^n y_n + T^n u_n\|.
\end{aligned}
$$

Thus, we have

$$\begin{aligned}
&\left(1+\alpha_n^2\right)\|x_{n+1}-u_{n+1}\| \\
&\quad\leqslant \left\{1-\left[1-(1+k_n)\alpha_n\right]\alpha_n\right\}\|x_n-u_n\|+(1+k_n)\alpha_n^3\|x_n-u_n-T^n y_n+T^n u_n\| \\
&\qquad+\alpha_n\|T^n x_{n+1}-T^n u_{n+1}-T^n y_n+T^n u_n\| \\
&\quad\leqslant \left\{1-\left[1-(1+k_n)\alpha_n\right]\alpha_n\right\}\|x_n-u_n\|+(1+k_n)\alpha_n^3\|u_n-T^n u_n\| \\
&\qquad+(1+k_n)\alpha_n^3\|x_n-T^n y_n\|+\alpha_n\|T^n x_{n+1}-T^n y_n\|+\alpha_n\|T^n u_{n+1}-T^n u_n\|.
\end{aligned} \tag{12}$$

But

$$\begin{aligned}
\|x_n-T^n y_n\| &\leqslant \|x_n-u_n\|+\|u_n-T^n u_n\|+\|T^n u_n-T^n y_n\| \\
&\leqslant \|x_n-u_n\|+\|u_n-T^n u_n\|+L\|u_n-y_n\|
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\|u_n-y_n\| &= \left\|(1-\beta_n)(u_n-x_n)+\beta_n(u_n-T^n x_n)\right\| \\
&\leqslant (1-\beta_n)\|u_n-x_n\|+\beta_n\|u_n-T^n x_n\| \\
&\leqslant (1-\beta_n)\|u_n-x_n\|+\beta_n\left[\|T^n u_n-T^n x_n\|+\|u_n-T^n u_n\|\right] \\
&\leqslant (1-\beta_n)\|u_n-x_n\|+\beta_n L\|u_n-x_n\|+\beta_n\|u_n-T^n u_n\| \\
&= (1-\beta_n+\beta_n L)\|u_n-x_n\|+\beta_n\|u_n-T^n u_n\| \\
&\leqslant L\|u_n-x_n\|+\beta_n\|u_n-T^n u_n\|,
\end{aligned} \tag{14}$$

because $1 \leqslant L \Rightarrow 1-\beta_n+\beta_n L \leqslant L$.

Substituting (14) into (13) we obtain

$$\begin{aligned}
\|x_n-T^n y_n\| &\leqslant \|u_n-x_n\|+\|u_n-T^n u_n\|+L\left(L\|u_n-x_n\|+\beta_n\|u_n-T^n u_n\|\right) \\
&\leqslant (1+L^2)\|u_n-x_n\|+(1+L\beta_n)\|u_n-T^n u_n\|,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\|T^n x_{n+1}-T^n y_n\| &\leqslant L\|x_{n+1}-y_n\| = L\left\|(1-\alpha_n)x_n+\alpha_n T^n y_n-y_n\right\| \\
&\leqslant L\left[(1-\alpha_n)\|x_n-y_n\|+\alpha_n\|T^n y_n-y_n\|\right].
\end{aligned} \tag{16}$$

Using (14),

$$\begin{aligned}
\|T^n y_n-y_n\| &\leqslant \|T^n y_n-T^n u_n\|+\|T^n u_n-u_n\|+\|y_n-u_n\| \\
&\leqslant L\|y_n-u_n\|+\|T^n u_n-u_n\|+\|y_n-u_n\| \\
&\leqslant (1+L)\|y_n-u_n\|+\|T^n u_n-u_n\| \\
&\leqslant (1+L)\left[L\|x_n-u_n\|+\beta_n\|T^n u_n-u_n\|\right]+\|T^n u_n-u_n\| \\
&= (1+L)L\|x_n-u_n\|+\left[(1+L)\beta_n+1\right]\|T^n u_n-u_n\|.
\end{aligned} \tag{17}$$

From (4) we have

$$\begin{aligned}
\|x_n-y_n\| &= \left\|x_n-(1-\beta_n)x_n-\beta_n T^n x_n\right\| = \beta_n\|x_n-T^n x_n\| \\
&\leqslant \beta_n\left[\|x_n-u_n\|+\|T^n u_n-u_n\|+\|T^n u_n-T^n x_n\|\right] \\
&\leqslant \beta_n\left[(1+L)\|x_n-u_n\|+\|T^n u_n-u_n\|\right].
\end{aligned} \tag{18}$$

Substituting (18) and (17) into (16), we obtain

$$\begin{aligned}
&\|T^n x_{n+1} - T^n y_n\| \\
&\quad \leqslant L\big[(1-\alpha_n)\|x_n - y_n\| + \alpha_n\|T^n y_n - y_n\|\big] \\
&\quad = L\big\{(1-\alpha_n)\beta_n\big[(1+L)\|x_n - u_n\| + \|T^n u_n - u_n\|\big] \\
&\qquad + \alpha_n(1+L)\big[L\|x_n - u_n\| + \beta_n\|u_n - T^n u_n\|\big] + \alpha_n\|u_n - T^n u_n\|\big\} \\
&\quad = \big\{L(1-\alpha_n)\beta_n(1+L) + \alpha_n(1+L)L^2\big\}\|x_n - u_n\| \\
&\qquad + \big\{\beta_n L(1-\alpha_n) + \alpha_n L\big[(1+L)\beta_n + 1\big]\big\}\|T^n u_n - u_n\|. \qquad (19)
\end{aligned}$$

Using (14) we have

$$\begin{aligned}
\|x_n - T^n y_n\| &\leqslant \|x_n - u_n\| + \|u_n - T^n u_n\| + \|T^n y_n - T^n u_n\| \\
&\leqslant \|x_n - u_n\| + \|u_n - T^n u_n\| + L\|y_n - u_n\| \\
&\leqslant \|x_n - u_n\| + \|u_n - T^n u_n\| + L\big[L\|x_n - u_n\| + \beta_n\|u_n - T^n u_n\|\big] \\
&= (1+L^2)\|x_n - u_n\| + (1+\beta_n L)\|u_n - T^n u_n\|. \qquad (20)
\end{aligned}$$

Substituting (19) and (20) into (12), and using the facts that $(1+\alpha_n^2)^{-1} \leqslant 1$, we get

$$\begin{aligned}
&\big(1+\alpha_n^2\big)\|x_{n+1} - u_{n+1}\| \\
&\quad \leqslant \big\{1 - \big[1 - (1+k_n)\alpha_n\big]\alpha_n\big\}\|x_n - u_n\| \\
&\qquad + (1+k_n)\alpha_n^3\big\{(1+L^2)\|x_n - u_n\| + (1+\beta_n L)\|u_n - T^n u_n\|\big\} \\
&\qquad + (1+k_n)\alpha_n^3\|u_n - T^n u_n\| + \alpha_n\|T^n u_{n+1} - T^n u_n\| \\
&\qquad + \alpha_n\big\{L(1-\alpha_n)\beta_n(1+L) + \alpha_n(1+L)L^2\big\}\|x_n - u_n\| \\
&\qquad + \alpha_n\big\{\beta_n L(1-\alpha_n) + \alpha_n L\big[(1+L)\beta_n + 1\big]\big\}\|u_n - T^n u_n\|, \qquad (21)
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leqslant \big\{1 - \big[1 - (1+k_n)\alpha_n\big]\alpha_n + (1+k_n)\alpha_n^3(1+L^2) \\
&\quad + \alpha_n\big\{L(1-\alpha_n)\beta_n(1+L) + \alpha_n(1+L)L^2\big\}\big\}\|x_n - u_n\| \\
&\quad + \big\{(1+k_n)\alpha_n^3(2+\beta_n L) \\
&\quad + \alpha_n\big\{\beta_n L(1-\alpha_n) + \alpha_n L\big[(1+L)\beta_n + 1\big]\big\}\big\}\|u_n - T^n u_n\| \\
&\quad + \alpha_n\|T^n u_{n+1} - T^n u_n\|. \qquad (22)
\end{aligned}$$

We may write

$$a_{n+1} \leqslant \gamma_n a_n + \sigma_n, \qquad (23)$$

where

$$\begin{aligned}
&a_n := \|x_n - u_n\|, \\
&\gamma_n := \big\{1 - \big[1 - (1+k_n)\alpha_n\big]\alpha_n + (1+k_n)\alpha_n^3(1+L^2) \\
&\qquad + \alpha_n\big\{L(1-\alpha_n)\beta_n(1+L) + \alpha_n(1+L)L^2\big\}\big\}, \\
&\sigma_n := \big\{(1+k_n)\alpha_n^3(2+\beta_n L) \\
&\qquad + \alpha_n\big\{\beta_n L(1-\alpha_n) + \alpha_n L\big[(1+L)\beta_n + 1\big]\big\}\big\}\|u_n - T^n u_n\| \\
&\qquad + \alpha_n\|T^n u_{n+1} - T^n u_n\|. \qquad (24)
\end{aligned}$$

We have

$$L(1-\alpha_n)\beta_n(1+L)+\alpha_n(1+L)L^2 \leqslant L(1+L)\big[(1-\alpha_n)\beta_n+\alpha_n L\big]$$
$$\leqslant L(1+L)[L\beta_n+\alpha_n L] = L^2(1+L)(\alpha_n+\beta_n).$$

The last inequality is true because $L \geqslant 1$. From (5) it follows that for all n sufficiently large we have

$$\alpha_n \leqslant \frac{1}{5}\sup\left(\left(\frac{1}{1+L^2}\right),\left(\frac{1}{1+k_n}\right)\right), \qquad \alpha_n+\beta_n \leqslant \frac{1}{5}\left(\frac{1}{(1+L)L^2}\right);$$

thus

$$\begin{aligned}\gamma_n &\leqslant 1-\big[1-(1+k_n)\alpha_n\big]\alpha_n+(1+k_n)\alpha_n^3(1+L^2)+\alpha_n L^2(1+L)(\alpha_n+\beta_n)\\ &\leqslant 1-\big[1-(1+k_n)\alpha_n\big]\alpha_n+\frac{1}{25}\alpha_n+\frac{1}{5}\alpha_n\\ &\leqslant 1-\big[1-(1+k_n)\alpha_n\big]\alpha_n+\frac{2}{5}\alpha_n\\ &\leqslant 1-\frac{4}{5}\alpha_n+\frac{2}{5}\alpha_n = 1-\frac{2}{5}\alpha_n. \end{aligned} \tag{25}$$

Thus $\gamma_n \leqslant 1-(2/5)\alpha_n$ for all n sufficiently large, from which we obtain relation (7),

$$a_{n+1} \leqslant (1-\lambda_n)a_n+\sigma_n. \tag{26}$$

The fact that Mann iteration (3) converges, i.e., $\lim_{n\to\infty} u_n = x^*$ (more precisely using $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$), it is easy to see that $\sigma_n = o(\lambda_n)$. All the assumptions from Lemma 2 are now satisfied, so $\lim_{n\to\infty} a_n = 0$. Hence,

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{27}$$

Since $\lim_{n\to\infty} u_n = x^*$, (27) and the inequality

$$\|x_n-x^*\| \leqslant \|x_n-u_n\|+\|u_n-x^*\| \to 0 \quad (n\to\infty) \tag{28}$$

lead to $\lim_{n\to\infty} x_n = x^*$. □

Because an asymptotically nonexpansive map is asymptotically pseudocontractive and Lipschitzian (see Remarks 2 and 4), from Theorem 8 we obtain the following result.

Corollary 9. *Let B be a closed convex subset of an arbitrary Banach space X and $(x_n)_n$ and $(u_n)_n$ defined by* (3) *and* (4) *with $(\alpha_n)_n$ and $(\beta_n)_n$ satisfying* (5). *Let T be an asymptotically nonexpansive self-map of B. Let x^* be the fixed point of T. If $u_0 = x_0 \in B$, then the following two assertions are equivalent*:

(i) *Mann type iteration* (3) *converges to $x^* \in F(T)$*;
(ii) *Ishikawa type iteration* (4) *converges to $x^* \in F(T)$.*

The following result is from [4].

Theorem 10 [4]. *Let B be a closed convex subset of an arbitrary Banach space X and let T be a Lipschitzian strongly pseudocontractive self-map of B. Let $x_1 = u_1$ and let $(x_n)_n$ and $(u_n)_n$ be the Mann and Ishikawa iterations* (*that is* (3) *and* (4) *without "n" at the exponent of T*)*, with $(\alpha_n)_n$, $(\beta_n)_n$ satisfying* (5)*. Then the following are equivalent*:

(i) *The Mann iteration converges strongly to x^**;
(ii) *The Ishikawa iteration converges strongly to x^*.*

Theorem 8 is the analog of Theorem 10 for asymptotically pseudocontractive operators.

Our theorems are also true for set-valued mappings, if such maps admit appropriate single-valued selections.

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